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The contact problem for a piecewise-homogeneous plane with a semi-infinite inclusion☆

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ABSTRACT

A piecewise-homogenous elastic plate, reinforced with a semi-infinite inclusion, which intersects the interface at a right angle and is loaded with shear forces is considered. The contact stresses along the contact line are determined and the behaviour of the contact stresses in the neighbourhood of singular points is established. Using methods of the theory of analytical functions and integral transformations the problem is reduced to a system of singular integro-differential equations on the semi-axis. The solution is presented in explicit form.

Exact or approximate solutions of static contact problems for different regions, reinforced with elastic fastenings or thin inclusions, and also coverings of variable stiffness have been considered previously, and the behaviour of the contact stresses at the ends of the contact line have been investigated as a function of the variation of the geometrical and physical parameters of these components.[1–8] The first fundamental problem was solved for a piecewise-homogeneous plane, when a crack of finite length reaches the interface of the two bodies at a right angle,[9] and also similar problems for a piecewise-uniform plane when acted upon by symmetrical normal stresses on the crack surfaces.[10,11]

In this paper we consider a piecewise-homogeneous elastic plate, reinforced with a semi-infinite inclusion and acted upon by a shear force with a strength of $\tau_k^0(x)$. With respect to the inclusion, which has the form of a thin slightly curved cover, it is assumed that it is rigidly bonded to the plate and is stretched or compressed as a rod which is in a uniaxial stress state. It is assumed that the horizontal deformations of the inclusion and the elastic piecewise-homogeneous continuous plate, loaded along the semi-axis with shear stresses, are compatible. There is no normal contact stress at the joint.

The problem consists of determining the jump of the contact shear stresses $\tau_k(x)$ along the contact line and in establishing their behaviour at singular points (i.e., in the neighbourhoods of the ends of the inclusion) and is formulated as follows: suppose the elastic body S occupies the plane of the complex variable $z=x+iy$, which, along the line $L=(-\infty,1)$, contains an elastic inclusion with modulus of elasticity $E_0(x)$, thickness $h_0(x)$ and Poisson's ratio ν_0 and consists of two half-planes of different materials

$$S_1 = \{z|\operatorname{Re}z>0,\ z\notin \bar{l}_1=[0,1]\},\quad S_2=\{z|\operatorname{Re}z<0,\ z\notin \bar{l}_2=]-\infty,0]\}$$

soldered along the $x=0$ axis. Quantities and functions, referred to the half-plane S_k, will be given the subscript k $(k=1,2)$, while the boundary values of other functions on the upper and lower edges of the inclusion will be given plus and minus superscripts, respectively (Fig. 1).

At the interface we have the condition's of continuity

$$\sigma_x^{(1)}=\sigma_x^{(2)},\quad \tau_{xy}^{(1)}=\tau_{xy}^{(2)},\quad u_1=u_2,\quad \upsilon_1=\upsilon_2 \tag{1}$$

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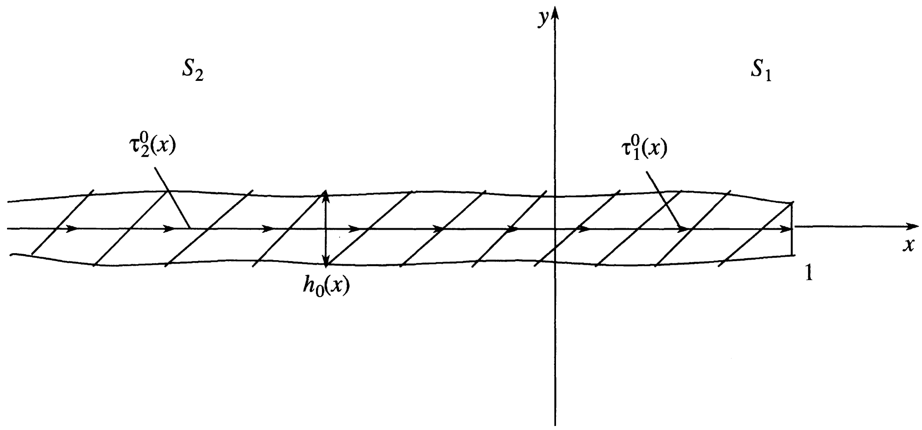

Fig. 1.

On the sections l_k we have the following conditions:

$$\frac{du_1^{(0)}(x)}{dx} = \frac{1}{E(x)}\left\{P_0 - \int_0^x [\tau_1(x) - \tau_1^0(t)]dt\right\}, \quad x \in l_1$$

$$\frac{du_2^{(0)}(x)}{dx} = \frac{1}{E(x)}\int_{-\infty}^x [\tau_2(t) - \tau_2^0(t)]dt, \quad x \in l_2; \quad E(x) = \frac{E_0(x)h_0(x)}{1-\nu_0^2} \tag{2}$$

Here $u_k^{(0)}(x)$ are the horizontal displacements of the points of inclusion, while the equilibrium conditions of the individual parts of the inclusion have the form

$$\int_{-\infty}^0 (\tau_2(t) - \tau_2^{(0)}(t))dt = P_0, \quad P_0 - \int_0^1 (\tau_1(t) - \tau_1^{(0)}(t))dt = P \tag{3}$$

where P_0 and P are the unknown axial forces at the points $x = 0$ and $x = 1$ respectively (Fig. 2).

To determine these forces we must add the following relations to conditions (3)

$$P_0 = \int_{-\frac{h_0(0)}{2}}^{\frac{h_0(0)}{2}} \sigma_x^{(1)}(0, y)dy, \quad P = \int_{-\frac{h_0(1)}{2}}^{\frac{h_0(1)}{2}} \sigma_x^{(1)}(1, y)dy \tag{4}$$

where $\sigma_x^{(1)}(x, y)$ is the horizontal component of the stress field in the half-plane S_1.

From the KolosovMuskhelishvili formulae.[12]

$$\varphi_k(z) = \overline{z\varphi_k'(z)} + \overline{\psi(z)} = i\int_{z_k}^z (X_n^{(k)} + iY_n^{(k)})ds + C_k \equiv R_k(z)$$

$$\aleph_k\varphi_k(z) - \overline{z\varphi_k'(z)} - \overline{\psi(z)} = 2\mu_k(u_k + i\upsilon_k), \quad \aleph_k = 3 - 4\nu_k$$

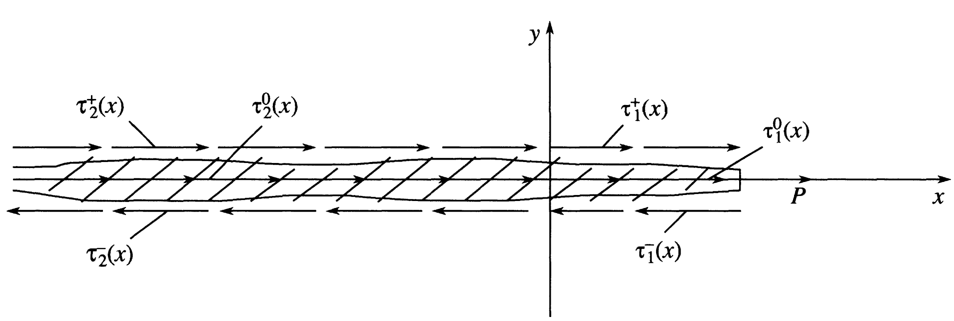

Fig. 2.

taking into account the conditions of continuity of the components of the displacements in the plate with the cover

$$u_k^+ = u_k^-, \quad \upsilon_k^+ = \upsilon_k^-$$

we obtain

$$\varphi_k^+(x) - \varphi_k^-(x) = \frac{i}{1+\aleph_k}\int_{x_k}^{x}\tau_k(t)dt \equiv if_k(t), \;\; \psi_k^+(x) - \psi_k^-(x) = -i(\aleph_k f_k(x) + xf'_k(x)),$$

$$x \in l_k$$

The solutions of these problems have the form

$$\varphi_k(z) = \frac{1}{2\pi}\int_{l_k}\frac{f_k(t)dt}{t-z} + W_k(z) = w_k(z) + W_k(z)$$

$$\psi_k(z) = -\frac{1}{2\pi}\int_{l_k}\frac{(\aleph_k f_k(t) + tf'_k(t))dt}{t-z} + Q_k(z) = q_k(z) + Q_k(z); \quad z \in S_k \tag{5}$$

where $W_k(z)$ and $Q_k(z)$ are analytic functions in the half-plane S_k.

By introducing the functions

$$\omega_k(z) = -z\varphi'_k(z) + \psi_k(z) = \eta_k(z) + \Omega_k(z) \tag{6}$$

where

$$\eta_k(z) = -zw'_k(z) + q_k(z), \quad \Omega_k(z) = -zW_k(z) + Q_k(z)$$

we reduce the KolosovMuskhelishvili formulae to the form

$$\varphi_k(z) + (z+\bar{z})\overline{\varphi'_k(z)} + \overline{\omega_k(z)} = R_k(z)$$

$$\aleph_k\varphi_k(z) - (z+\bar{z})\overline{\varphi'_k(z)} - \overline{\omega_k(z)} = 2\mu_k(u_k + i\upsilon_k)$$

Writing conditions (1) in terms of the functions (6), and acting on the equalities obtained with the singular operator

$$S(\cdot) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{(\cdot)}{t-z}dt, \quad z \in S_k$$

we obtain, with respect to the functions $W_1(z)$, $\Omega_1(z)$, $\overline{W_2(-\dot{z})}$, $\overline{\Omega_2(-\dot{z})}$, a system of four equations, the solution of which has the form

$$W_1(z) = e_1\overline{\eta_1(-\bar{z})} + r_2w_2(z), \quad \Omega_1(z) = h_1\overline{w_1(-\bar{z})} + m_2\eta_2(z)$$

$$\overline{W_2(-\bar{z})} = e_2\eta_2(z) + r_1\overline{w_1(-\bar{z})}, \quad \overline{\Omega_2(-\bar{z})} = h_2w_2(z) + m_1\overline{\eta_1(-\bar{z})}$$

Using these relations, we obtain from formulae (5) and (6)

$$\varphi_k(z) = \frac{1}{2\pi}\int_{l_k}\left[\frac{1}{t-z} - \frac{e_k\aleph_k}{t+z} + \frac{e_kz}{(t+z)^2}\right]f_k(t)dt - \frac{e_k}{2\pi}\int_{l_k}\frac{tf'_k(t)}{t+z}dt + \frac{r_{3-k}}{2\pi}\int_{l_{3-k}}\frac{f_{3-k}(t)}{t-z}dt$$

$$\psi_k(z) = \frac{1}{2\pi}\int_{l_k}\left[\frac{-\aleph_k}{t-z} + \frac{h_k}{t+z} + \frac{e_k(1+\aleph_k)z}{(t+z)^2} - \frac{2e_kz^2}{(t+z)^3}\right]f_k(t)dt +$$

$$+\frac{1}{2\pi}\int_{l_k}\left[\frac{-1}{t-z} + \frac{e_kz}{(t+z)^2}\right]tf'_k(t)dt +$$

$$+\frac{1}{2\pi}\int_{l_{3-k}}\left[\frac{-m_3\aleph_{3-k}}{t-z} + \frac{(r_{3-k}-m_{3-k})z}{(t-z)^2}\right]f_{3-k}(t)dt - \frac{m_{3-k}}{2\pi}\int_{l_{3-k}}\frac{tf'_{3-k}(t)}{t-z}dt; \quad k = 1, 2 \tag{7}$$

where

$$e_k = \frac{\mu_1-\mu_2}{\Delta_k},\ r_k = \frac{\mu_{3-k}(\aleph_k+1)}{\Delta_{3-k}},\ m_k = \frac{\mu_k(\aleph_{3-k}+1)}{\Delta_{3-k}},\ h_k = \frac{\aleph_k\mu_{3-k}-\aleph_{3-k}\mu_k}{\Delta_{3-k}}$$
$$\Delta_k = \aleph_k\mu_{3-k}+\mu_k$$

Introducing expressions (7) into the equality

$$u_k(z) = \frac{1}{2\mu_k}\mathrm{Re}[\aleph_k\varphi_k(z)-z\overline{\varphi_k'(z)}-\overline{\psi_k(z)}]$$

and taking the limit as $z\to x\pm i0$, we obtain a system of singular integro-differential equations

$$\frac{E(x)}{4\pi\mu_1}\int_0^\infty\left[\frac{2\aleph_1}{t-x}+\frac{\aleph_1^2e_1+h_1}{t+x}-\frac{4e_1t^2}{(t+x)^3}\right]\tilde{f}_1'(t)dt-$$
$$-\frac{E(x)}{4\pi\mu_1}\int_0^\infty\left[\frac{r_2\aleph_1+m_2\aleph_2}{t+x}+\frac{2(m_2-r_2)t}{(t+x)^2}\right]\tilde{f}_2'(t)dt = \begin{cases}-(1+\aleph_1)\tilde{f}_1(x)+T_1(x),\ x\in(0,1)\\ E(x)u_1'(x),\quad x\in(1,\infty)\end{cases}$$
$$-\frac{E(x)}{4\pi\mu_2}\int_0^\infty\left[\frac{2\aleph_2}{t-x}+\frac{\aleph_2^2e_2+h_2}{t+x}-\frac{4e_2t^2}{(t+x)^3}\right]\tilde{f}_2'(t)dt+\frac{E(x)}{4\pi\mu_2}\int_0^\infty\left[\frac{r_1\aleph_2+m_1\aleph_1}{t+x}+\frac{2(m_1-r_1)t}{(t+x)^2}\right]\tilde{f}_1'(t)dt =$$
$$(1+\aleph_2)\tilde{f}_2(x)+T_2(x),\ x\in(0,\infty) \tag{8}$$

where

$$\tilde{f}_1(x) = \begin{cases}f_1(x),\quad x\in(0,1)\\ 0,\quad x\in(1,\infty)\end{cases},\quad \tilde{f}_2(x) = f_2(-x)$$

The functions $T_1(x)$ and $T_2(x)$ depend on the known functions $\tau_k^{(0)}(x)(k=1.2)$ and on the unknown constants P_0 and P, i.e.,

$$T_1(x) = \int_0^x\tau_1^0(t)dt-g_1(x),\quad T_2(x) = \int_{-x}^0\tau_2^0(-t)dt-g_2(x)$$
$$g_1(x) = \frac{E(x)}{2\mu_1}((\aleph_1-1)\alpha(x)-x\alpha'(x)-\delta(x))$$
$$g_2(x) = \frac{E(x)}{2\mu_2}((\aleph_2-1)\beta(-x)-x\beta'(-x)-\gamma(-x))$$

where

$$\alpha(x) = C_1\left\{\frac{e_1}{(1+x)^2}+\frac{1}{1-x}+\frac{e_1\aleph_1}{1+x}\right\}+C_2\frac{r_2}{x},\quad \beta(x) = C_1\frac{r_1}{1-x}+C_2\frac{(1-e_2\aleph_2)}{x}$$

$$\gamma(x) = C_1\left\{\frac{m_1\aleph_1}{1-x}+\frac{m_1-r_1}{(1-x)^3}\right\}+C_2\frac{h_2-\aleph_2}{x}$$

$$\delta(x) = C_1\left\{\frac{\aleph_1}{1-x}+\frac{h_1}{1+x}-\frac{e_1(1+\aleph_1)}{(1+x)^2}-\frac{2e_1x}{(1+x)^3}\right\}-C_2\frac{m_2\aleph_2}{x}$$

$$C_1 = \frac{T_1^0+P_0-P}{2\pi(1+\aleph_1)},\quad C_2 = \frac{P_0+T_2^0}{2\pi(1+\aleph_2)},\quad T_1^0 = \int_0^1\tau_1^0(t)dt,\quad T_2^0 = \int_{-\infty}^0\tau_2^0(t)dt$$

To solve system (8), when the stiffness of the inclusion varies linearly, i.e., $E(x) = h|x|$, $x \in (-\infty, 1)$, making the replacement of variables $t = e^{\zeta}$, $x = e^{\xi}$ in system (8) and using a Fourier transformation,[13] we obtain the system

$$G_1(s)F^-(s) + \breve{G}_2(s)\Phi(s) = -(1+\aleph_1)F^-(s) + \Psi^+(s) + P_1(s)$$

$$G_2(s)\Phi(s) + \breve{G}_1(s)F^-(s) = (1+\aleph_2)\Phi(s) + P_2(s), \quad s = s_0 - i\varepsilon, \quad \varepsilon > 0 \tag{9}$$

where

$$F^-(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\tilde{f}_1(e^{\xi})e^{i\xi z}d\xi, \quad \Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\tilde{f}_2(e^{\xi})e^{i\xi z}d\xi$$

$$\Psi^+(z) = \frac{1}{\sqrt{2\pi}}\int_{0}^{+\infty}u'(e^{\xi})e^{i\xi z}d\xi, \quad P_1(z)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}T_1(e^{\xi})e^{i\xi z}d\xi, \quad P_2(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}T_2(e^{\xi})e^{i\xi z}d\xi$$

$$G_k(z) = \frac{1}{\operatorname{sh}\pi z}[2\aleph_k \operatorname{ch}\pi z + \aleph_k^2 e_k + h_k - 2e_k i z^2(z+i)]$$

$$\breve{G}_k(z) = \frac{1}{\operatorname{sh}\pi z}[(r_k\aleph_{3-k} + m_k\aleph_k) + 2(m_k - r_k)z^2]$$

Eliminating the function

$$\Phi(s) = [P_2(s) - \breve{G}_1(s)F^-(s)]/\Delta(s); \quad \Delta(s) = G_2(s) - (1+\aleph_2) \tag{10}$$

from system (9), we obtain

$$G(s)F^-(s) = \Psi^+(s) + H(s) \tag{11}$$

where

$$G(s) = G_1(s) + (1+\aleph_1) - \breve{G}_1(s)\breve{G}_2(s)/\Delta(s), \quad H(s) = P_1(s) - P_2(s)\breve{G}_2(s)/\Delta(s)$$

It can be shown that

$$G(s) \to 3\aleph_1 + 1 \equiv \alpha \ \text{ as } \ t \to +\infty, \quad G(s) \to 1 - \aleph_1 \equiv \beta \ (\beta < 0) \ \text{ as } \ t \to -\infty$$

$$G(t) = \frac{t^2 g_0(t)}{\operatorname{sh}\pi t}, \quad g_0(t) > 0$$

Condition (11) can be represented in the form

$$\frac{\sqrt{1+t^2}}{t}G(t)\frac{tF^-(t)}{\sqrt{t-i}} = \Psi^+(t)\sqrt{t+i} + H(t)\sqrt{t+i} \tag{12}$$

where $\sqrt{z+i}$ and $\sqrt{z-i}$ mean the branches, analytic in the planes cut along the rays, drawn from the points $z = -i$ and $z = i$ in the x direction, and which obtain positive and negative values respectively on the upper side of the cut. By virtue of this choice of the branches the function $\sqrt{1+z^2}$ becomes analytic in the strip $-1 < \operatorname{Im} z < 1$ and takes a positive value on the real axis. Hence, the problem can be formulated as follows: it is required to obtain the function $\Psi^+(z)$, holomorphic in the $\operatorname{Im} z > 0$ half-plane and which vanishes at infinity, and the function $F^-(z)$, holomorphic in the half-plane $\operatorname{Im} z < 0$, apart from points that are roots of the function $G(z)$, which vanish at infinity and are continuous on the real axis by condition (12).

The solution of the problem has the form

$$F^-(z) = \frac{\sqrt{z-i}\,X(z)}{z}\left\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\sqrt{t+i}\,H(t)dt}{X^+(t)(t-z)} + \frac{c}{z-i}\right\}, \quad \operatorname{Im} z < 0$$

$$\Psi^+(z) = \frac{X(z)}{\sqrt{z+i}}\left\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\sqrt{t+i}\,H(t)dt}{X^+(t)(t-z)} + \frac{c}{z-i}\right\}, \quad \operatorname{Im} z > 0$$

$$F^-(z) = \{\Psi^+(z) + H(z)\}/G(z), \quad 0 < \operatorname{Im} z < 1 \tag{13}$$

where

$$X(z) = \exp\left\{\frac{z+i}{2\pi i}\int_{-\infty}^{+\infty}\frac{\ln G_0(t)dt}{(t+i)(t-z)}\right\}, \quad G_0(t) = \frac{\sqrt{1+t^2}}{t}G(t)$$

The constant c is found from the condition $F^{-}(0)=0(1)$. We obtain

$$c = \frac{H(0)}{2\sqrt{i}X^{+}(0)} + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{H(t)\sqrt{t+i}dt}{X^{+}(t)t}$$

From formulae (13) we obtain, by an inverse Fourier transformation, expressions for the required functions $f_k(x)$, after simple operations we find $\sigma_x^{(1)}(x, y)$ and, using formulae (4), we obtain relations for P_0 and P. (In this case $P_0=0$.)

It can be shown that $F^{-}(x+i0)=F^{-}(x-i0)$ and, consequently, the function $F^{-}(z)$ is holomorphic in the half-plane, apart from points which are zeros of the function $G(z)$ in the upper half-plane.

We will investigate the behaviour of the contact stresses in the neighbourhood of the singular points $z=0$ and $z=1$. It can be shown that, in the first equality of (13),

$$F^{-}(x) = c_0/\sqrt{x-i} + F_0^{-}(x)$$

where $F_0^{-}(x)$ is the Fourier transform of the function $f_0(x)$, continuous on the semi-axis $x \le 0$, apart, maybe, from the point $x=0$, at which it may have a logarithmic singularity. We obtain by an inverse transformation

$$\tau_1(x) = O(1/\sqrt{1-x}), \quad x \to 1-$$

We will now investigate the behaviour of the function $\tau_1(x)$ in the neighbourhood of the point $z=0$. The poles of the function $F^{-}(z)$ in the region $D_0=\{z{:}0<\operatorname{im}z<1\}$ may be zeros of the function

$$g(z) = (G_1(z)+(1+\aleph_1))(G_2(z)-(1+\aleph_2))-\breve{G}_1(z)\breve{G}_2(z)$$

We will assume that $i\tau_0$ is the simple zero, of least modulus, the function $g(z)$ in the region D_0. Then, applying Cauchy's theorem on residues to the function $e^{-i\xi z}F^{-}(z)$ for the rectangle $D(N)$ with boundary $L(N)$, which consists of the sections

$$[-N, N],\ [N+i0, N+i\beta_0],\ [N+i\beta_0, -N+i\beta_0],\ [-N+i\beta_0, -N+i0]$$

$\tau_0 < \beta_0 < \tau_0^1$, $g(i\tau_0^1 = 0)$ we obtain

$$\int_{L(N)} F^{-}(t)e^{-i\xi t}dt = \int_{-N}^{N}F^{-}(t)e^{-i\xi t}dt - e^{\beta_0\xi}\int_{-N}^{N}F^{-}(t+i\beta_0)e^{-i\xi t}dt + \rho(N, \xi) = K_1e^{\xi\tau_0} \tag{14}$$

where $\rho(N,\xi) \to 0$ as $N \to \infty$. Taking the limit in equality (14) and reverting to the old variables, we obtain

$$\tau_1(x) = (1+\aleph_1)f_1'(x) = (1+\aleph_1)K_1x^{\tau_0-1} + O(x^{\beta_0-1}), \quad x \to 0+$$

Similarly, by determining the function $\Phi(t)$ from (10) and carrying out an inverse Fourier transformation, we obtain after calculations

$$\tau_2(x) = (1+\aleph_2)f_2'(x) = K_2x^{\mu_0-1} + O(x^{\tau_0-1}), \quad x \to 0-$$

where $i\mu_0(\mu_0<\tau_0)$ is the simplest to zero, of least modulus, of the function $\Delta(z) = G_2(z)-(1+\aleph_2)\text{в}$ in the region D_0. If $\mu_0>\tau_0$ then $\tau_2(x) = O(x^{\tau_0-1})$, $x \to 0-$.

If the functions $g(z)$ and $\Delta(z)$ have no simple zeros in the region D_0, the contact stresses may have a logarithmic-type singularity at the origin of coordinates (if, for example, the point

z = i is a double zero of the function $g(z)$ or $\Delta(z)$).

It should be noted that the system of integro-differential Eq. (8) obtained reduces to a single equation in the following interesting cases.

Case 1. The semi-infinite inclusion has a constant stiffness or the stiffness varies as $hx^{\omega}(\omega \neq 1, h=\text{const})$ and the inclusion reaches the interface of the two materials. Then, by a Fourier transformation of the required function one obtains a boundary-value problem of the theory of analytic functions with a shift for a strip (a Carleman type problem).

Case 2. The stiffness of the semi-infinite or finite inclusion varies linearly and the inclusion reaches the interface of the two materials. Then, by a Fourier transformation of the required function one obtains an algebraic equation or a boundary-value problem of linear matching respectively.

The use of methods of the theory of analytic functions and of integral transformations enables effective solutions of the above problems to be obtained.

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